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Improved Two-Point Function Approximations for Design Optimization

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Two-point approximations are developed by utilizing both the function and gradient information of two data points. The objective of this work is to build a high-quality approximation to realize computational savings in solving complex optimization and reliability analysis problems. Two developments are proposed in the new two-point approximations: 1) calculation of a correction term by matching with the previous known function value and supplementing it to the first-order approximation for including the effects of higher order terms and 2) development of a second-order approximation without the actual calculation of second-order derivatives by using an approximate Hessian matrix. Several highly nonlinear functions and structural examples are used for demonstrating the new two-point approximations that improved the accuracy of existing first-order methods.

I. Introduction

DEVELOPMENTof new and improved constraint approximations has been an active area of research in the mathematical optimization community over the last two decades.[1–3] More accurate function approximations reduce the finite element analysis cost and increase the search domain in each iteration of an optimization run. Recent developments in constraint approximations addressed methods for eigenvalues, dynamic response, static forces, structural reliability, etc. Some of the typical methods are summarized in the following. Kirsch[4] developed a scaling factor for a stiffness matrix and approximated the displacements, stresses, and forces with respect to sizing and topology variables. Vanderplaats and Salajegheh[5] used two levels of approximations for discrete sizing and shape design of structures, first for solving the continuous problem and next the discrete one using a dual theory. Canfield[6] developed an approach for eigenvalues by approximating the modal strain and kinetic energies independently. Thomas et al.[7] presented an approximation for the frequency response of damped structures by using the concept of Ref. 6. A multivariate spline approximation was developed by Wang and Grandhi[8] making use of the previously generated exact finite element analyses. Similarly, a Hermite approximation for n-dimensional problems used several data points and had the property of reproducing the function and gradient values at the known data points.[9] Canfield[10] used multipoint data in building an approximate Hessian matrix and constructed a second-order approximation for improving the accuracy. Toropov et al.[11] approximated functions as a linear combination or products of variables, and the coefficients of the polynomial were computed by applying a least squared approach on multiple data points.

Fadel et al.[12] considered intermediate variables in terms of exponentials and they were computed by matching the approximate function gradients with the previous point's exact values. This approach did not compare the function values of the previous point. Wang and Grandhi[13] used a single exponential for all of the variables and calculated it by matching the approximate function with the previous point exact value, and the gradient information of the previous point was not utilized. The proposed paper develops an improved two-point approximation utilizing both function and gradient information of two data points. In the new two-point approximation, two developments are proposed: 1) calculation of a correction term based on two function values and supplementing it to the first-order approximation to include the effects of higher order terms and 2) development of a second-order approximation using diagonal elements of the Hessian matrix. Both of these approximations improved the accuracy of the existing first-order methods and demonstrated their validity for larger changes in design variables. Several highly nonlinear functions and structural examples are used for demonstrating this new two-point approximation.

II. First-Order Approximation Methods

In this section, several one-point and two-point approximation methods are reviewed, and a new two-point approximation is introduced. The known data points are denoted as $X_1(x_{1,1}, x_{2,1}, \ldots, x_{n,1})$ and $X_2(x_{1,2}, x_{2,2}, \ldots, x_{n,2})$ where the function and gradient information is available. Here n is the number of variables. The function to be approximated is denoted as $g(X)$. The one-point approximations are expanded at the first point X_1 using the function and derivatives at this point. The two-point approximations are expanded at the current point X_2 and use the values of the function and/or derivatives at two points. Mathematical details of these approximations are described next.

A. One-Point Approximations

1. *Linear Approximation*

The linear approximation is a first-order Taylor series expansion at X_1

$$\tilde{g}(X) = g(X_1) + \sum_{i=1}^{n} \frac{\partial g(X_1)}{\partial x_i}(x_i - x_{i,1}) \tag{1}$$

where x_i is the ith component of design variables X and $x_{i,1}$ is the ith component of the known point X_1. This approximation is very popular since the function and its derivatives are needed in search direction calculation and no additional computation is involved in developing the approximate function.

2. *Reciprocal Approximation*

The reciprocal approximation is the first-order Taylor series expansion in the reciprocals of the variables $y_i = 1/x_i (i = 1, 2, \ldots, n)$. It can be written in terms of the original variables, x_i,

$$\tilde{g}(X) = g(X_1) + \sum_{i=1}^{n} \frac{\partial g(X_1)}{\partial x_i}(x_i - x_{i,1})\left(\frac{x_{i,1}}{x_i}\right) \tag{2}$$

This approximation was proven to be very efficient for truss structures with stress and displacement constraints because of its scaling property.

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3. Generalized Power Approximation

This approximation was presented by Prasad,[14] who defined generalized variables as

$$\phi_k = \begin{cases} 1/(p-1)t_k^{1-p} & \text{if} \quad p \neq 1 \\ -\ln[t_k] & \text{if} \quad p = 1 \end{cases} \qquad k = 1, 2, \ldots, m \quad (3)$$

where ϕ_k represents the relationship between the constraint functions and the physical parameters (such as thickness, areas, etc.), p can be any real number, t_k is the kth physical parameter, which can be written in a general form as

$$t_k(X) = t_{k0} + \sum_{i=1}^{n} x_i f_{ik}, \qquad k = 1, 2, \ldots, m \tag{4}$$

where t_{k0} represents a reference parameter value, x_i is a design variable, n is the number of design variables, m is the number of independent physical parameters, and f_{ik} represents the matrix of coefficients linking design variables to a group.

Using Eq. (3) and the first-order Taylor series expansion, the functions can be approximated using function values and their derivatives at X_1,

$$\tilde{g}(X) = g(X_1) + \frac{1}{p-1} \sum_{i=1}^{n} \frac{\partial g(X_1)}{\partial x_i} \sum_{k=1}^{m} \left[t_k^{1-p}(X) - t_k^{1-p}(X_1) \right] \bigg/ \frac{\partial \phi_k}{\partial x_i} \qquad \text{for } p \neq 1 \tag{5a}$$

$$\tilde{g}(X) = g(X_1) - \sum_{i=1}^{n} \frac{\partial g(X_1)}{\partial x_i} \sum_{k=1}^{m} \ln\left[\frac{t_k(X)}{t_k(X_1)}\right] \bigg/ \frac{\partial \phi_k}{\partial x_i} \qquad \text{for } p = 1 \tag{5b}$$

$$\frac{\partial \phi_k(X_1)}{\partial x_i} = -t_k^{-p} f_{ik} \tag{5c}$$

where p was arbitrarily selected based on the nature of the function being approximated. For example, p was selected as 1 for the linear approximation and as -1 for the reciprocal approximation. This generalized power approximation (GPA) has the advantage of using different forms in different constraints, problem types, and finite element models, but the selection of the approximate forms is not determined automatically by itself and requires experience and knowledge.

B. Two-Point Approximations

1. Modified Reciprocal Approximation

One problem of the reciprocal approximation given in Eq. (2) is that it becomes unbounded when one of the variables approaches zero. Therefore, a modified approximation was presented in Ref. 15,

$$\tilde{g}_m(X) = g(X_2) + \sum_{i=1}^{n} \frac{\partial g(X_2)}{\partial x_i}(x_i - x_{i,2})\left(\frac{x_{mi} + x_{i,2}}{x_{mi} + x_i}\right) \tag{6}$$

where X_2 is the current point. The values of x_{mi} were evaluated by matching the derivatives at the previous point X_1, that is,

$$\frac{\partial g(X_1)}{\partial x_i} = \left(\frac{x_{mi} + x_{i,2}}{x_{mi} + x_{i,1}}\right)^2 \frac{\partial g(X_2)}{\partial x_i} \tag{7}$$

or

$$x_{mi} = [(x_{i,2} - \eta_i x_{i,1}/(\eta_i - 1)] \tag{8}$$

where

$$\eta_i^2 = \frac{\partial g(X_1)}{\partial x_i} \bigg/ \frac{\partial g(X_2)}{\partial x_i} \tag{9}$$

When the ratio of the derivatives is negative, the derivatives at the previous point X_1 were not matched. In that case x_{mi} is set to a very large number, so that the linear approximation is used for the ith variable.

2. Two-Point Posynomial Approximation

The approximation is of the form

$$\tilde{g}(X) = C \prod_{i=1}^{n} x_i^{a_i} \tag{10}$$

A least squares formulation was adopted to determine the coefficient C and exponents a_i ($i = 1, 2, \ldots, n$), in which the minimization problem is written as

$$E = \frac{1}{2}\left\{\left[g(X_2) - C\prod_{i=1}^{n} x_{i,2}^{a_i}\right]^2 + \left[g(X_1) - C\prod_{i=1}^{n} x_{i,1}^{a_i}\right]^2 + \sum_{j=1}^{n}\left[\frac{\partial g(X_2)}{\partial x_j} - C\prod_{i=1}^{k} x_{i,2}^{a_i} \cdot \ln(x_j)\right]^2\right\} \tag{11}$$

In this posynomial approximation,[16] the value of the function and its gradients at the current point and the value of the function at the previous point are used in determining the unknown constants of Eq. (10).

3. Two-Point Exponential Approximation

In Ref. 12, the authors combined the methods of Ref. 14 presented in Sec. II.A.3 and Ref. 15 presented in Sec. II.B.1, and developed a two-point exponential approximation. It is a linear Taylor approximation in terms of the intervening variables

$$y_i = x_i^{p_i}, \qquad i = 1, 2, \ldots, n \tag{12}$$

where the exponent p_i for each design variable was evaluated by matching the derivatives of the approximate function with the previous data point gradients, that is,

$$\frac{\partial g(X_1)}{\partial x_i} = \left(\frac{x_{i,1}}{x_{i,2}}\right)^{p_i - 1} \cdot \frac{\partial g(X_2)}{\partial x_i}, \qquad i = 1, 2, \ldots, n \tag{13}$$

From this equation, p_i is computed as

$$p_i = 1 + \left\{\ln\left[\frac{\partial g(X_1)}{\partial x_i} \bigg/ \frac{\partial g(X_2)}{\partial x_i}\right] \bigg/ \ln[x_{i,1}/x_{i,2}]\right\} \tag{14}$$

The approximate function is given in terms of the original variables x_i as

$$\tilde{g}(X) = g(X_2) + \sum_{i=1}^{n}\left(\frac{x_{i,2}^{1-p_i}}{p_i}\right)\frac{\partial g(X_2)}{\partial x_i}\left(x_i^{p_i} - x_{i,2}^{p_i}\right) \tag{15}$$

In this approximation, the value of p_i is limited from -1 to $+1$. In fact, this algorithm has a better adaptability for different structural problems if the limitation of p_i is removed. Therefore, a two-point exponential approximation (TPEA) without p_i limitation is considered in this paper, which is called the TPEA-change method.

4. Two-Point Adaptive Nonlinear Approximation

This approximation based on two-point information was presented in Ref. 13 using adaptive intervening variables. The intervening variables were defined as

$$y_i = x_i^r, \qquad i = 1, 2, \ldots, n \tag{16}$$

where r represents the nonlinearity index, which is different at each iteration but is the same for all variables. The nonlinearity index was determined by matching the function value of the previous design point; that is, r is numerically calculated such that the difference of the exact and approximate $g(X)$ at the previous point X_1 becomes zero,

$$g(X_1) - \left\{g(X_2) + \frac{1}{r}\sum_{i=1}^{n} x_{i,2}^{1-r}\frac{\partial g(X_2)}{\partial x_i}\left(x_{i,1}^r - x_{i,2}^r\right)\right\} = 0 \tag{17}$$

where r can be any positive or negative real number (not equal to zero). The two-point adaptive nonlinear approximation (TANA) is

$$\tilde{g}(X) = g(X_2) + \frac{1}{r}\sum_{i=1}^{n} x_{i,2}^{1-r}\frac{\partial g(X_2)}{\partial x_i}\left(x_i^r - x_{i,2}^r\right) \tag{18}$$

This approximation was extensively used in truss, frame, plate, and turbine blade structural optimization and probabilistic design. The results presented in Refs. 9, 10, and 17 demonstrated the accuracy and adaptive nature of building a nonlinear approximation.

C. Proposed Two-Point Approximations

In the described methods of Secs. II.B.3 and II.B.4, the TPEA method matched the derivatives of exact and approximate function values at the previous point for evaluating p_i, whereas TANA matched only the function values of exact and approximate calculations at the previous point for finding r. The TPEA method used the derivative values at two points and the function value at the current point, whereas TANA used the function values at two points and the derivative values at the current point. To utilize more information in constructing a better approximation, the proposed method combines these two methods and produces an improved approximation. In this paper, two improved methods are presented; one is called new TANA-1 and the other one is called new TANA-2. In these methods, both the function and derivative values of two points are utilized in developing the approximation.

1. New Two-Point Adaptive Nonlinear Approximation-1

In this method, the intervening variables given in Eq. (12) are used. The approximate function is assumed as

$$\tilde{g}(X) = g(X_1) + \sum_{i=1}^{n}\frac{\partial g(X_1)}{\partial x_i}\frac{x_{i,1}^{1-p_i}}{p_i}\left(x_i^{p_i} - x_{i,1}^{p_i}\right) + \varepsilon_1 \tag{19}$$

where ε_1 is a constant, representing the residue of the first-order Taylor approximation in terms of the intervening variables y_i ($y_i = x_i^{p_i}$). Unlike the other two-point approximations, this approximation is expanded at the previous point X_1 instead of the current point X_2. The reason is that if the approximation was constructed at X_2 the approximate function value would not be equal to the exact function value at the expanding point because of the correction term ε_1. In actual optimization, to obtain more accurate predictions closer to the current point, X_1 is selected as the expansion point. The approximate function and its derivative values are matched with the current point.

By differentiating Eq. (19), the derivative of the approximate function with respect to the ith design variable x_i is written as

$$\frac{\partial \tilde{g}(X)}{\partial x_i} = \left(\frac{x_i}{x_{i,1}}\right)^{p_i-1}\frac{\partial g(X_1)}{\partial x_i}, \qquad i = 1, 2, \ldots, n \tag{20}$$

From this equation, p_i can be evaluated by letting the exact derivatives at X_2 equal the approximation derivatives at this point, that is,

$$\frac{\partial g(X_2)}{\partial x_i} = \frac{\partial \tilde{g}(X_2)}{\partial x_i} = \left(\frac{x_{i,2}}{x_{i,1}}\right)^{p_i-1}\frac{\partial g(X_1)}{\partial x_i}, \qquad i = 1, 2, \ldots, n \tag{21}$$

where p_i can be any positive or negative real number (not equal to zero). Equation (21) has n equations and n unknown constants. It is easy to solve because each equation has a single unknown constant p_i. Here, a simple adaptive search technique is used to solve them. The numerical iteration for calculating each p_i starts from $p_i = 1$. When p_i is increased or decreased by a step length (0.1), the error between the exact and approximation derivatives at X_2 is calculated. If this error is smaller than the initial error (e.g., corresponding to $p_i = 1$), the iteration is repeated until the allowable error (0.001) or limitation of p_i is reached, and p_i is determined. Otherwise, the step length of p_i is decreased by half and the iteration process is repeated until the final p_i is obtained. This search is computationally inexpensive because Eq. (21) is available in a closed form and very easy to implement.

Equation (21) matches only the derivative values of the current point, and so a difference between the exact and approximate function values at the current point may exist. This difference is eliminated by adding the correct term ε_1 in the approximation. The term ε_1 is computed by matching the approximate and exact function values at the current point,

$$\varepsilon_1 = g(X_2) - \left\{g(X_1) + \sum_{i=1}^{n}\frac{\partial g(X_1)}{\partial x_i}\frac{x_{i,1}^{1-p_i}}{p_i}\left(x_{i,2}^{p_i} - x_{i,1}^{p_i}\right)\right\} \tag{22}$$

where ε_1 is a constant during a particular iteration. This TANA-1 method is simple and, more importantly, the new approximation function and derivative values are equal to the exact values at the current point.

2. New Two-Point Adaptive Nonlinear Approximation-2

This improved method also uses the intervening variables given in Eq. (12). The approximation is written by expanding the function at X_2:

$$\tilde{g}(X) = g(X_2) + \sum_{i=1}^{n}\frac{\partial g(X_2)}{\partial x_i}\frac{x_{i,2}^{1-p_i}}{p_i}\left(x_i^{p_i} - x_{i,2}^{p_i}\right) + \frac{1}{2}\varepsilon_2\sum_{i=1}^{n}\left(x_i^{p_i} - x_{i,2}^{p_i}\right)^2 \tag{23}$$

This approximation is a second-order Taylor expansion in terms of the intervening variables y_i ($y_i = x_i^{p_i}$), in which the Hessian matrix has only diagonal elements of the same value ε_2. Therefore, this approximation does not need the calculation of the second-order derivatives. Unlike the original second-order approximation, this approximation is expanded in terms of the intervening variables y_i, so the error from the approximate Hessian matrix is partially corrected by adjusting the nonlinearity index p_i. In contrast to the true quadratic approximation, this approximation is closer to the actual function for highly nonlinear problems because of its adaptability. Equation (23) has $n + 1$ unknown constants, and so $n + 1$ equations are required. Differentiating Eq. (23), n equations are obtained by matching the derivatives with the previous point, X_1:

$$\frac{\partial g(X_1)}{\partial x_i} = \left(\frac{x_{i,1}}{x_{i,2}}\right)^{p_i-1}\frac{\partial g(X_2)}{\partial x_i} + \varepsilon_2\left(x_{i,1}^{p_i} - x_{i,2}^{p_i}\right)x_{i,1}^{p_i-1}p_i \qquad i = 1, 2, \ldots, n \tag{24}$$

Another equation is obtained by matching the exact and approximate function values with the previous point X_1, that is,

$$g(X_1) = g(X_2) + \sum_{i=1}^{n}\frac{\partial g(X_2)}{\partial x_i}\frac{x_{i,2}^{1-p_i}}{p_i}\left(x_{i,1}^{p_i} - x_{i,2}^{p_i}\right) + \frac{1}{2}\varepsilon_2\sum_{i=1}^{n}\left(x_{i,1}^{p_i} - x_{i,2}^{p_i}\right)^2 \tag{25}$$

There are many algorithms to solve these $n + 1$ equations as simultaneous equations. Again, a simple adaptive search technique is used to solve them. First, ε_2 is fixed at a small initial value (0.5), the numerical iteration described in Sec. II.C.1 is used to solve each p_i, and the differences between the exact and approximate function and derivative values at X_1 are calculated. Then, ε_2 is increased or decreased by a step length (0.1), p_i, and the differences between the exact and approximate function and derivative values at X_1 are recalculated. If these differences are smaller than the initial error (e.g., corresponding to $\varepsilon_2 = 0.5$), the iteration is repeated until the allowable error (0.001) or limitation of ε_2 is reached, and the optimum combination of ε_2 and p_i is determined.

In this new TANA-2 method, the exact function and derivative values are equal to the approximate function and derivative values, respectively, at the previous point.

III. Numerical Examples

Several examples are selected to examine the accuracy of the proposed approximation, and the results are compared with the linear approximation given in Sec. II.A.1, the reciprocal approximation given in Sec. II.A.2, the two-point exponential approximation with p_i limitation and two-point exponential approximation (TPEA-change) without p_i limitation given in Sec. II.B.3, and the two-point adaptive nonlinear approximation given in Sec. II.B.4. The relative error is calculated as follows:

$$\text{relative error} = \frac{\text{exact} - \text{approximation}}{\text{exact}} \quad (26)$$

The examples include three explicit functions and an implicit displacement constraint of a 313-member frame structure requiring a finite element analysis. In all of the examples, the test points are derived using

$$X = X_0 + \alpha D$$

where X_0 is an initial point, which is defined as the expanding point for all of the approximations except for TANA-1. It is defined as the matching point for TANA-1. Here α is a step length, and $\boldsymbol{D}$ is a direction vector which is selected as $\boldsymbol{D} = \{1, 1, 1, 1, \ldots\}^T$ for case 1, $\boldsymbol{D} = \{-1, 1, -1, 1, \ldots\}^T$ for case 2, $\boldsymbol{D} = \{1, 0, 1, 0, \ldots\}^T$ for case 3, and $\boldsymbol{D} = \{0, 1, 0, 1, \ldots\}^T$ for case 4.

A. Example 1

This example is taken from Ref. 18, and the constraint function is defined as

$$g(X) = \frac{10}{x_1} + \frac{30}{x_1^3} + \frac{15}{x_2} + \frac{2}{x_2^3} + \frac{25}{x_3} + \frac{108}{x_3^3} + \frac{40}{x_4} + \frac{47}{x_4^3} - 1.0$$

All of the approximations except TANA-1 are expanded at the point $X_2(1, 1, 1, 1)$, and the previous point is selected as $X_1(1.2, 1.2, 1.2, 1.2)$. TANA-1 is expanded at X_1 and matched with the values at X_2. The relative errors of several methods are plotted in Figs. 1a, 1b, 1c, and 1d for the four cases. Figure 1a shows that when design variables are changed along the same direction as X_1 and X_2 points (case 1), TANA-2 has the best accuracy (the absolute values of relative errors are smaller than 7% everywhere), and TANA-1, TANA, and TPEA-change have almost similar accuracy (errors are smaller than 20% everywhere). All of these approximations (TANA, TANA-1, TANA-2, and TPEA-change) have lower errors (almost zero) when $\alpha > 0$ because the second point lies on the right side of the α axis. For the other three cases, the second point X_1 does not lie in the same direction as $\boldsymbol{D}$. When design variables x_1 and x_3 are changed along an opposite direction of x_2 and x_4 (case 2), TANA-2 has the smallest errors when $\alpha > 0$, whereas TANA has the best accuracy when $\alpha < 0$. TANA-1 and TPEA-change also work well and have the same results where the errors are smaller than 9% everywhere. When the design variables are changed only along x_1 and x_3 (case 3), TANA-2 has very small errors (smaller than 4%), and TPEA-change and TANA-1 have almost the same accuracy as TANA-2. When the design variables are changed only along x_2 and x_4 (case 4), TANA-2, TANA-1, and TPEA-change are accurate when $\alpha > 0$, and TANA has the lowest errors when $\alpha < 0$. For all four cases, the relative errors of linear, reciprocal, and TPEA approximations are large, and TPEA has the same results as the reciprocal approximation because its exponents p_i are equal to -1. TPEA-change substantially improved the accuracy of TPEA because the limitations of the exponents were removed. In the TPEA-change method, the exponents p_i are -2.7625, -1.5, -2.825, and -2.4875 for x_1, x_2, x_3, and x_4, respectively. TANA has a single nonlinearity index r which is equal to -2.7. TANA-1 has the same exponents p_i as the TPEA-change and ε_1 is -0.0862. The exponents for x_1, x_2, x_3, and x_4 in TANA-2 method are -2.7375, -1.45, -2.825, and -2.475, respectively, and ε_2 is 0.5527.

For case 1 of this example, the results comparisons with different expanding points for TANA-1 and different matching points for TANA-2 are plotted in Figs. 1e and 1f, respectively. Figure 1e shows that as the expanding point is changed from $X_2(0.5, 0.5, 0.5, 0.5)$ to $X_2(1.5, 1.5, 1.5, 1.5)$, the relative errors become positive from negative in the right side of the matching point, whereas the errors become negative from positive in the left side of the matching point for the TANA-1. When the expanding point is not too far from the matching point [between $X_2(0.7, 0.7, 0.7, 0.7)$ and $X_2(1.3, 1.3, 1.3, 1.3)$], the errors decrease in the right side while they increase in the left side as the expanding points move from left to right. However, when the expanding point is too far from the matching point, such as $X_2(0.5, 0.5, 0.5, 0.5)$, $X_2(1.5, 1.5, 1.5, 1.5)$ or even farther, then the errors in both sides of the α axis get worse. Also, the errors at the points between the expanding and the matching points increase gradually. Figure 1f shows that when the matching point is selected from the $X_1(0.9, 0.9, 0.9, 0.9)$ to the $X_1(1.3, 1.3, 1.3, 1.3)$ range for the TANA-2, the errors decrease in the right side of the α axis, whereas the errors increase in the left side of the α axis. However, when the matching point is selected as $X_1(1.5, 1.5, 1.5, 1.5)$, the errors in both sides of the α axis get worse because the matching point is far from the expanding point. When the matching point is selected as $X_1(0.5, 0.5, 0.5, 0.5)$ and $X_1(0.7, 0.7, 0.7, 0.7)$, the errors almost equal zero between the expanding point and the matching point, but the errors increase faster outside of this area. Figures 1e and 1f indicate that the accuracy of TANA-1 and TANA-2 methods are different when the expanding or matching point is changed, but this difference is not large if the expanding and matching points are not too far apart.

B. Example 2

This example is also taken from Ref. 18, and the constraint function is defined as

$$\begin{aligned} g(X) = {} & 180x_1 + 20x_2 - 3.1x_3 + 0.2x_4 - 5x_1x_2 + 37x_1x_3 \\ & + 8.7x_2x_4 - 3x_3x_4 - 0.1x_1^2x_2 + 0.001x_2^2x_3 + 95x_1x_4^2 \\ & - 81x_4x_3^2 + x_1^3 - 6.2x_2^3 + 0.48x_3^3 + 22x_4^3 - 1.0 \end{aligned}$$

All of the approximations except TANA-1 are expanded at the point $X_2(1, 1, 1, 1)$, and the previous point is defined as $X_1(0.8, 0.8, 0.8, 0.8)$. TANA-1 is expanded at X_1 and the values are matched at X_2. The relative errors of several methods are plotted in Figs. 2a and 2b for case 3 and case 4, respectively. Figure 2a shows that when the design variables are changed only along x_1 and x_3 (case 3), TANA-1, TANA-2, TPEA-change, TPEA, and linear approximation have errors smaller than 9% (absolute value) everywhere. TANA performed worse than the linear approximation. The reason is that the matching point used for determining r does not lie in the test direction, so the r value of 1.6625 is not able to perform better than the linear approximation at those test points far away from the matching point. But TANA-1, TANA-2, and TPEA-change approximations still improved the linear approximation accuracy even when the matching point is far from the test points because of different exponents p_i used for each design variable. When the design variables are changed along x_2 and x_3 (case 4), TANA-1, TANA-2, and TPEA-change have very small errors, and TANA has lower than 10% error everywhere. They are better than the linear, TPEA, and reciprocal approximations everywhere. For both cases, the relative errors of reciprocal approximation are large, and TPEA has similar results to the linear approximation because its exponents p_i are equal to 1.0 for case 1, and equal to 1.0, -1.0, 1.0, and 1.0 for the design variables x_1, x_2, x_3, and x_4 for case 2. TPEA-change substantially improved the accuracy of TPEA because of no exponent limitations, in which the exponents p_i are 1.65, -2.5255, 3.2375, and 2.9625 for the four variables. The nonlinearity index r of TANA is 1.6625. TANA-1 has the same exponents p_i as the TPEA-change, and ε_1 is -0.1183. The exponents for x_1, x_2, x_3, and x_4 in TANA-2 method are 1.6375, -2.9, 3.2625, and 2.95, respectively, and ε_2 is 0.4076.

C. Example 3

This example is highly complex and nonlinear and it is given as

$$\begin{aligned} g(X) = {} & 10x_1x_2^{-1}x_4^2x_6^{-3}x_7^{0.125} + 15x_1^{-1}x_2^{-2}x_3x_4x_5^{-1}x_7^{-0.5} \\ & + 20x_1^{-2}x_2x_4^{-1}x_5^{-2}x_6 + 25x_1^2x_2^2x_3^{-1}x_5^{0.5}x_6^{-2}x_7 \end{aligned}$$

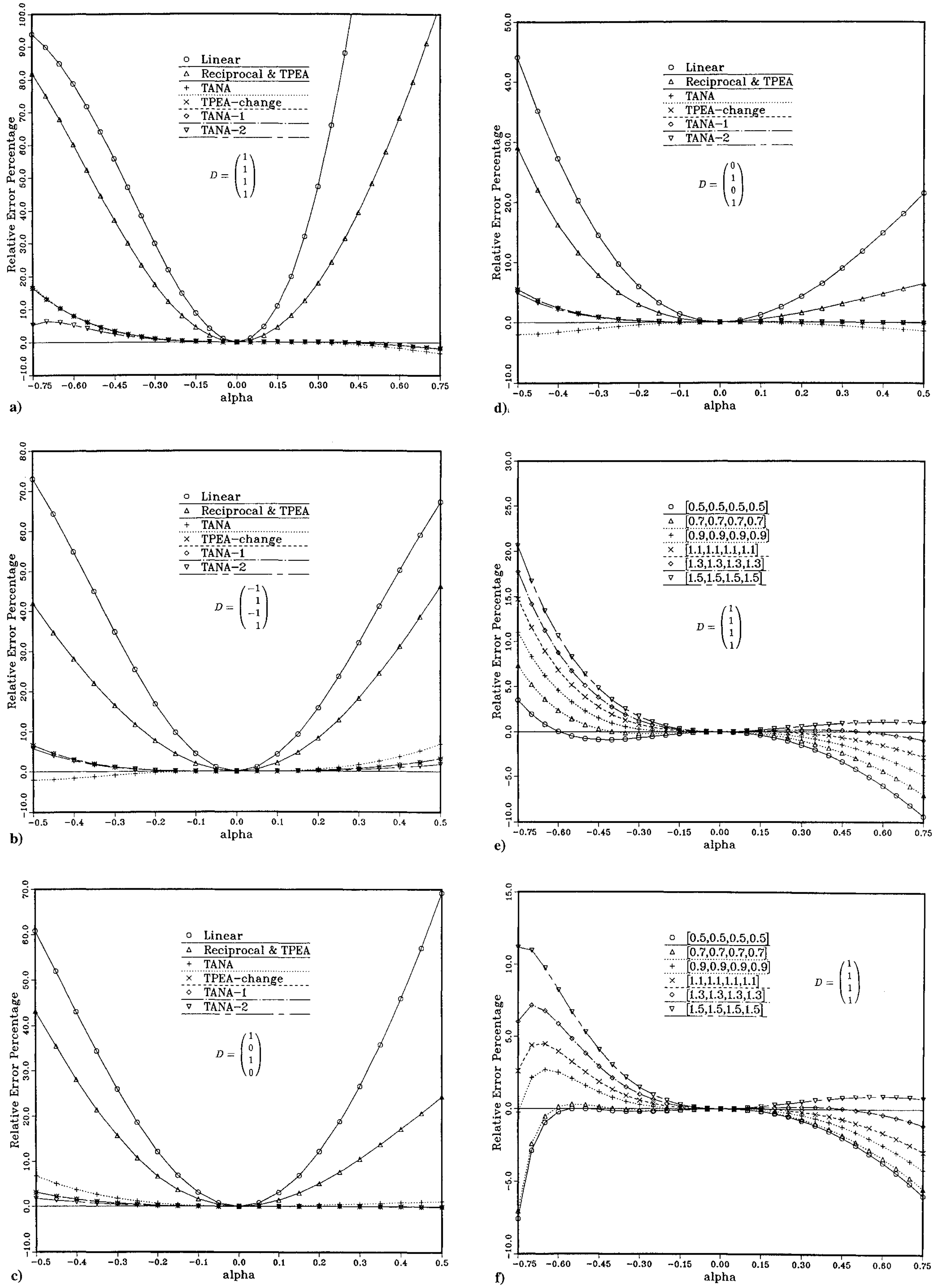


Fig. 1 Example 1: a) case 1, b) case 2, c) case 3, d) case 4, e) case 1 TANA-1 method, and f) case 1 TANA-2 method.

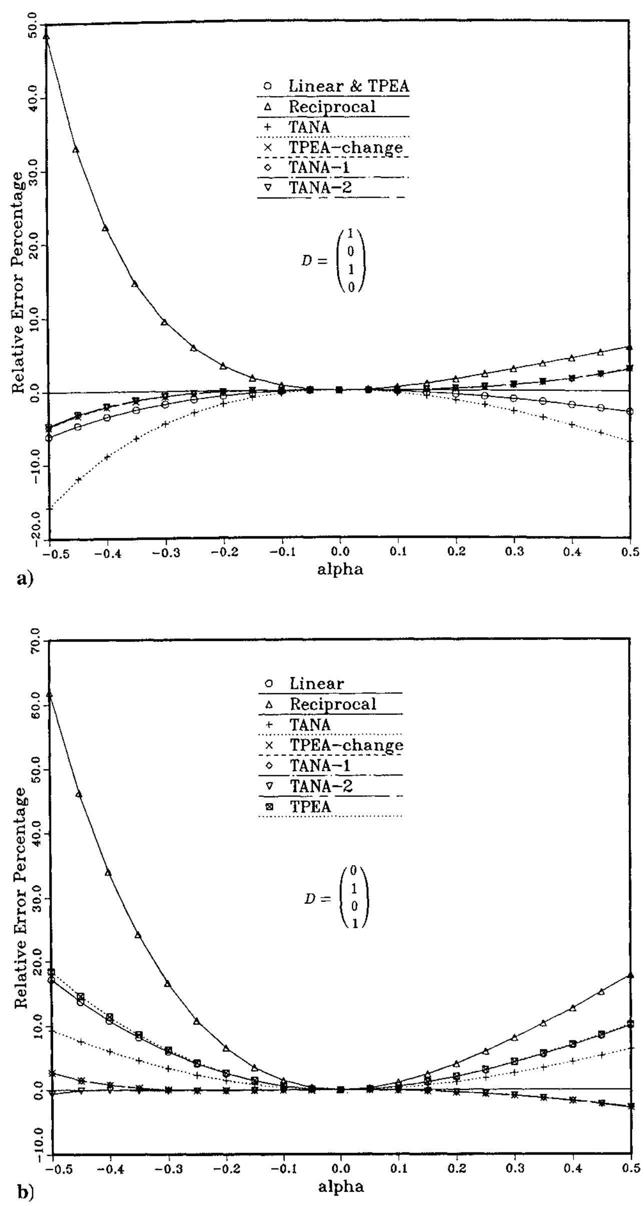


Fig. 2 Example 2: a) case 3 and b) case 4.

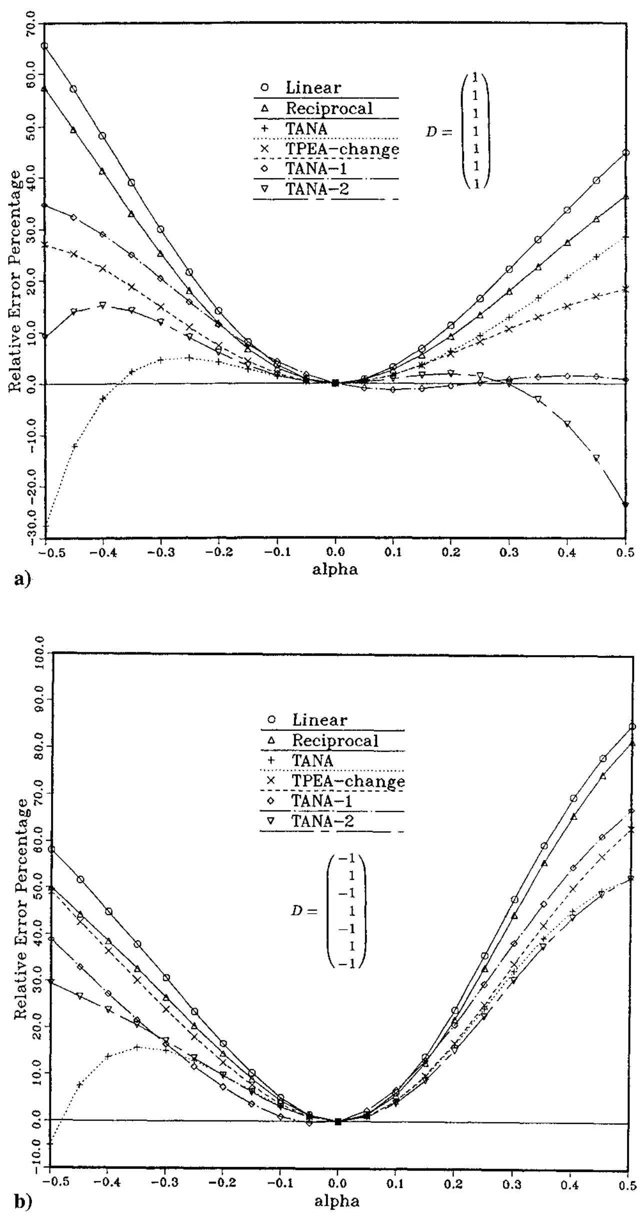


Fig. 3 Example 3: a) case 1 and b) case 2.

All of the approximations except TANA-1 are expanded at the point $X_2(1, 1, \ldots, 1)$, and the previous point is selected as $X_1(1.1, 1.1, \ldots, 1.1)$. TANA-1 is expanded at X_1 and matched at X_2. The relative errors of several methods are plotted in Figs. 3a and 3b for case 1 and case 2, respectively. Figure 3a shows that when the design variables are changed along the same direction (case 1), the errors of TANA are the lowest for most of the points when $\alpha < 0$, whereas TANA-1 is almost accurate when $\alpha > 0$. TANA-2 also works well, and its errors are lower than 20% (absolute value) almost everywhere. Figure 3b shows that when the design variables x_1 and x_3 are changed along the opposite direction of x_2 and x_4 (case 2), TANA-2 has the best accuracy when $\alpha > 0$. In the left side of the α axis, since the TANA error curve is nonmonotonic, it has the lowest errors at some points far away from the matching point. TANA-1 and TPEA-change are better than the linear and reciprocal methods. For both cases, the exponents of TPEA-change are 4.5, 4.4, 4.5, 0.318, −3.5, 2.488, and 3.813. The nonlinearity index r for TANA is −5.0. TANA-1 has the same exponents as TPEA-change, and ε_1 is 0.8233. The exponents of TANA-2 are 4.5, 3.25, 1.438, 0.126, −3.0, 3.013, and 2.638 for the variables, and ε_2 is 2.9523.

D. Example 4

The last example has an implicit constraint function requiring a finite element analysis. The frame structure shown in Fig. 4 is modeled with 313 beam elements having an I section. The cross-sectional areas of all members are selected as the design variables. The vertical loads at nodes 15, 16, 88, and 89 are −26, −30, −18, and −20 kips, respectively; the horizontal loads at nodes 6, 11, and 17–65 by 3, 68–82 by 7, and 90–175 by 5 are 4 kips; the horizontal load at node 1 is 2 kips. The approximation to the vertical displacement d at the tip point (node 16) is examined, which is an implicit function of 313 design variables and is written as

$$g(X) = d/d_{\text{lim}}$$

where d_{lim} is the displacement limit of 4.0 in. Young's modulus is 2.9×10^7 psi and Possion's ratio is 0.3. The initial cross-sectional areas X_0 are 30.59 in.^{2}, and the normalized design variables are defined as

$$\bar{X} = X/X_0$$

All of the approximations except TANA-1 are expanded at the point $X_2(1, 1, \ldots, 1)$, and the previous point is selected as $X_1(1.25, 1.25, \ldots, 1.25)$ for case 1 and $X_1(1.2, 0.8, 1.2, 0.8, \ldots, 1.2)$ for case 2. TANA-1 is expanded at X_1 and matched at X_2 for the two cases. The results comparisons of several methods are shown in Figs. 5a and 5b. Figure 5a shows that when the design variables are

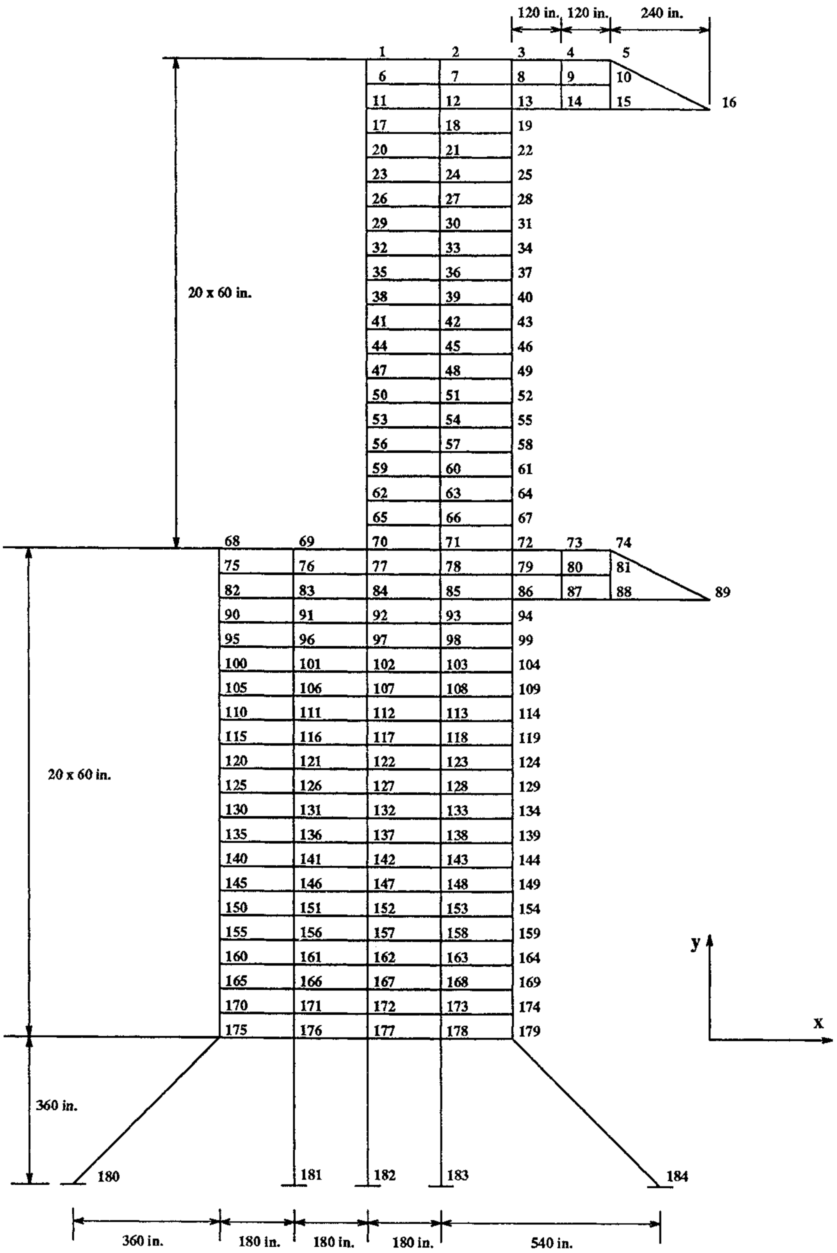
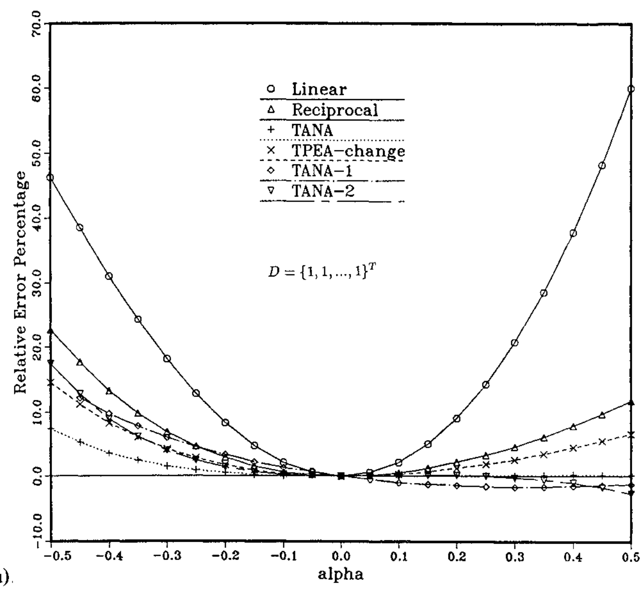
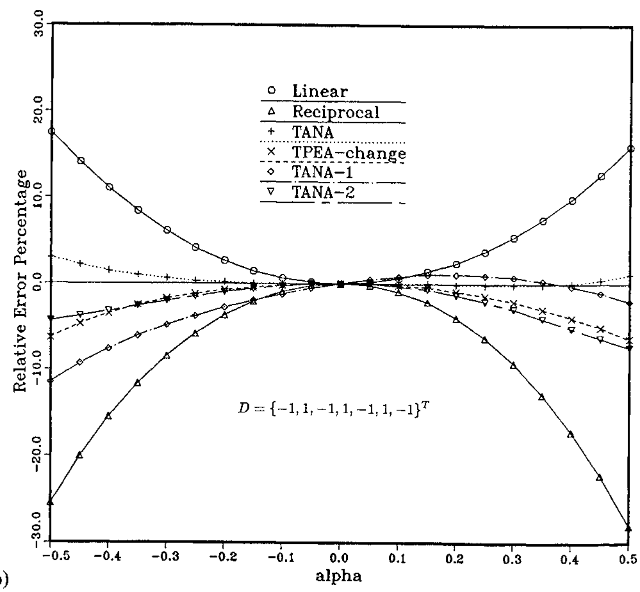


Fig. 4 Example 4, 313 member frame.



a)



b)

Fig. 5 Example 4, 313 member frame: a) case 1 and b) case 2.

changed along the same direction, the errors of TANA, TANA-1, and TANA-2 are smaller than linear and reciprocal approximations for case 1, particularly since when α is greater than zero the relative error is almost zero. Figure 5b shows that when the design variables are changed along $D = \{-1, 1, -1, 1, \ldots, -1\}^T$ (case 2), TANA has the lowest errors, and TANA-1, TANA-2, and TPEA-change also have good accuracy; they are much better than the linear and reciprocal methods. The results indicate that the proposed approximations have very good accuracy even for large-scale problems with hundreds of design variables.

Summary

Improved two-point approximations are developed in this work for the mathematical optimization and probability analysis of structures. Both function and derivative values at two points are used to construct the approximation. The function value and derivatives of the constructed approximation are matched with the exact values at the previous point as closely as possible. In the proposed approximations, the intervening variables with a different nonlinearity index p_i for each design variable are used to make the approximations closer to the actual nonlinear functions. The numerical calculation of exponent values is computationally inexpensive because the equations are available in a closed form.

The computational results indicate that the TANA-1, TANA-2, TANA, and TPEA-change methods can provide better accuracy than the linear, reciprocal, and TPEA methods for highly nonlinear problems where the functional dependency on design variables is difficult to predict. TANA-2 is better than TANA, TANA-1, and TPEA-change for most cases; whereas the accuracy improvement of TANA-1 compared to TANA-2 is not clearly evident, and it is better than TANA and TPEA-change only for a few cases. TANA works very well for the complex problems (the last two examples) even with a single nonlinearity index. The new approximations did extremely well in extrapolating a function. These adaptive two-point approximations are very effective for large-scale structures.

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